

# PHYS 232: Ruchardt's Experiment

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## Introduction

Consider an ideal gas in a closed container. The equation of state of the ideal gas relates the thermodynamic properties of volume  $V$ , pressure  $p$ , and temperature  $T$ :

$$pV = Nk_{\text{B}}T, \quad (1)$$

where  $N$  is the number of gas particles (atoms or molecules depending on the gas) and  $k_{\text{B}}$  is Boltzmann's constant. The gas particles collide with the container walls and exert a pressure which is related to the average kinetic energy of the gas particles:

$$pV = \frac{1}{3}Nm\overline{v^2}, \quad (2)$$

where  $m$  is the mass of each gas particle and  $\overline{v^2}$  is the mean square speed of the gas particles. Equating Eqns. 1 and 2 one arrives at:

$$\frac{1}{2}m\overline{v^2} = \frac{3}{2}k_{\text{B}}T, \quad (3)$$

which relates the kinetic energy of the gas particles to the thermal energy of the gas. In an ideal gas in a large container there is no preferred direction and it must be the case that:

$$\overline{v_x^2} = \overline{v_y^2} = \overline{v_z^2} = \frac{1}{3}\overline{v^2}. \quad (4)$$

That is, the mean square speed of a group of gas particles is the same in all three directions. Therefore, we obtain the following result:

$$\frac{1}{2}mv_i^2 = \frac{1}{2}k_{\text{B}}T, \quad (5)$$

where  $i$  can be any of  $x$ ,  $y$ , or  $z$ .

## Degrees of Freedom (DOF)

Each independent quantity that must be specified to determine the energy of a particle is called a *degree of freedom*. For a monatomic gas (one atom - helium, neon, argon, ...), there are only three translational degrees of freedom. The energy of each atom in the gas is specified by the average translational kinetic energy in each of the three possible directions:  $\overline{v_x^2}$ ,  $\overline{v_y^2}$ , and  $\overline{v_z^2}$ . From the arguments above, we have already seen that each of these degrees of freedom contributes equally to the total energy of the monatomic gas (a consequence of Eq. 4). This result is an example of the *equipartition of energy* - the energy of the gas is divided equally among the available degrees of freedom. The total internal energy  $U$  of the gas is:

$$U = \frac{f}{2} N k_B T, \quad (6)$$

where  $f$  is the number of DOF. For a monatomic gas  $f = 3$ .

## Rigid Diatomic Ideal Gas

Now consider a rigid diatomic gas (two atoms form a single molecule that resembles a dumbbell - H<sub>2</sub>, N<sub>2</sub>, O<sub>2</sub>, ...). Here, rigid implies that the interatomic distance between the two atoms is fixed. Figure 1 shows that now, in addition to the three translational DOF, each diatomic molecule has two rotational degrees of freedom (one rotation about the  $y$ -axis and one about the  $z$ -axis). The rotational kinetic energy is determined by:

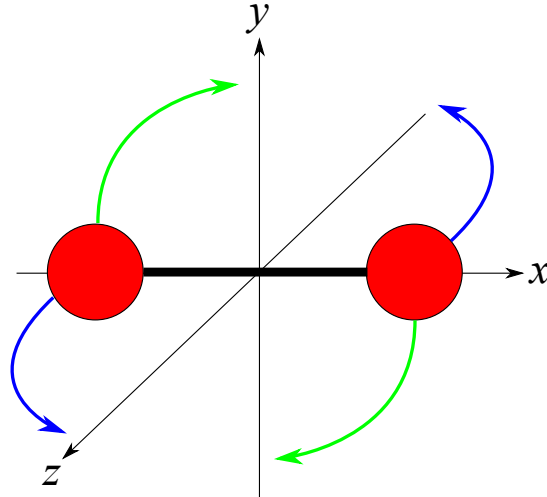
$$K_{\text{rot}} = \frac{1}{2} I \omega^2. \quad (7)$$

Note that there can be a rotation about the  $x$ -axis, but in this case the moment of inertia  $I$  is so small that this rotation can be ignored. The rigid diatomic ideal gas therefore has a total of five DOF and, assuming that the equipartition of energy still holds (which it does), the total internal energy of the gas is given by:

$$U = \frac{f}{2} N k_B T = \frac{5}{2} N k_B T. \quad (8)$$

## Diatomic Ideal Gas and Vibrations

In general a real diatomic gas is not rigid. Imagine instead that now the two atoms are connected by a spring. In this case there will be additional vibrational DOF that must be taken into account. However, quantum mechanically it turns out that these vibrational degrees of freedom are typically



rotation about y-axis

rotation about z-axis

Figure 1: The rigid diatomic molecule has two rotational degrees of freedom.

only activated at relatively high temperatures ( $T \gtrsim 2000$  K for  $O_2$ ). As a result, the rigid diatomic ideal gas is often a good approximation at room temperatures.

### Specific Heats and $\gamma$

We have seen that the internal energy of an ideal gas is given by:

$$U = \frac{f}{2} N k_B T = \frac{f}{2} n R T, \quad (9)$$

where  $n = N/N_A$  is the number of moles of gas,  $R = N_A k_B$  is the gas constant, and  $N_A = 6.02 \times 10^{23} \text{ mol}^{-1}$  is Avogadro's constant. The internal energy of the gas per mole is then:

$$u = \frac{U}{n} = \frac{f}{2} R T. \quad (10)$$

By definition the specific heat per unit mole at constant volume is given by:

$$c_v = \left( \frac{\partial u}{\partial T} \right)_v, \quad (11)$$

which for an ideal gas results in:

$$c_v = \frac{f}{2}R. \quad (12)$$

Finally we will make use of the relation  $c_p - c_v = R$  which holds for an ideal gas and where  $c_p$  is the specific heat per unit mole at constant pressure. Finally, we can then express the ratio of the specific heats as:

$$\frac{c_p}{c_v} \equiv \gamma = \frac{f+2}{f}. \quad (13)$$

The purpose of this laboratory is to **experimentally measure  $\gamma$  for air** and hence **determine the number of degrees of freedom of the gas particles making up our atmosphere.**

### Pre-lab Assignment

To understand Ruchardt's experiment we need to develop one more expression. To increase the internal energy of an ideal gas, resulting in a increased temperature, requires that you do work on the system:

$$W = dU = \frac{f}{2}nRdT, \quad (14)$$

where we have made use of Eq. 9. The work is equal to  $-pdV$  such that:

$$-pdV = \frac{f}{2}nRdT. \quad (15)$$

We can also make use of the equation of state  $pV = nRT$  to express:

$$dp = \frac{nR}{V}dT - \frac{nRT}{V^2}dV. \quad (16)$$

### Question 1

Use Eqn's 13, 15, 16, and the equation of state of an ideal gas to show that:

$$\frac{dp}{p} = -\gamma \frac{dV}{V}. \quad (17)$$

### Question 2

Integrate Eq. 17 to show that:

$$pV^\gamma = K, \quad (18)$$

where  $K$  is a constant.

## Measuring $\gamma$

Figure 2 shows the experimental setup used to measure  $\gamma$ . When the ball at the top of the cylinder

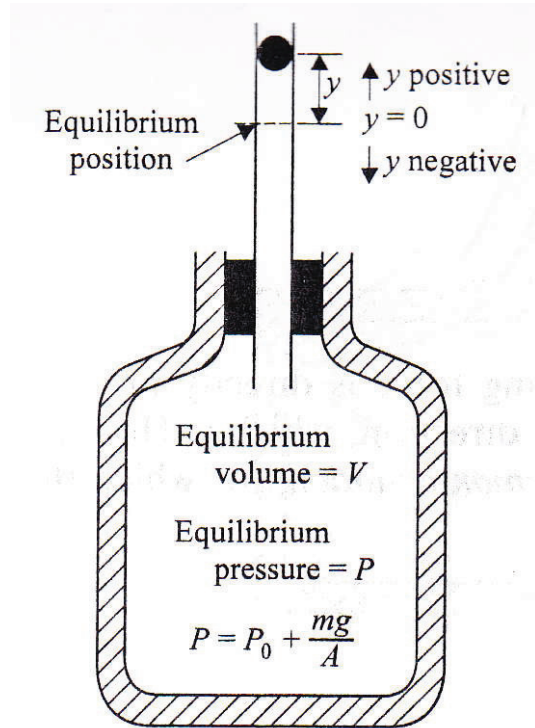


Figure 2: Experimental setup used to measure  $\gamma$ .

is released it will cause small changes to both the pressure and volume of the gas inside the large container. We want to relate those changes to each other. Start by taking the differential of Eq. 18:

$$V^\gamma dp + \gamma p V^{\gamma-1} dV = 0. \quad (19)$$

The pressure is force per unit area such that its magnitude can be written  $dp = F/A = m\ddot{y}/A$  where  $m$  is the mass of the ball,  $A$  is the cross-sectional area of the cylindrical tube, and  $\ddot{y}$  is the acceleration of the ball. The change in volume is  $dV = yA$ . Making these substitutions in Eq. 19 leads to:

$$m\ddot{y} = -\frac{\gamma p A^2}{V} y. \quad (20)$$

You should recognize this expression as simple harmonic motion (eg. the spring force  $F = m\ddot{x} = -kx$ ) which results in oscillations at an angular frequency of  $\omega = \sqrt{k/m}$ . Identifying  $\gamma p A^2/V$  as

the “ $k$ ” in Eq. 20 leads to oscillations with a period of:

$$\tau = 2\pi\sqrt{\frac{mV}{\gamma pA^2}}, \quad (21)$$

where  $\tau$  is used to denote the period rather than  $T$  which is reserved for temperature.

Use the experimental setup of Fig. 2 to determine  $\gamma$  for air. Release the ball from rest and time the period of the oscillations. The oscillations will be damped (decreasing amplitude) due to friction between the ball and the walls of the cylinder, however the period of the oscillations will remain approximately constant. Each time you release the ball, you should be able to observe several complete oscillations. Repeat the measurement many many times to determine an average and standard error. What is the number of DOF for air particles (you need to do proper error calculations!)? Does the result agree with your expectations?

Finally, determine  $\gamma$  more accurately using the experimental setup shown in Fig. 3. Here, the

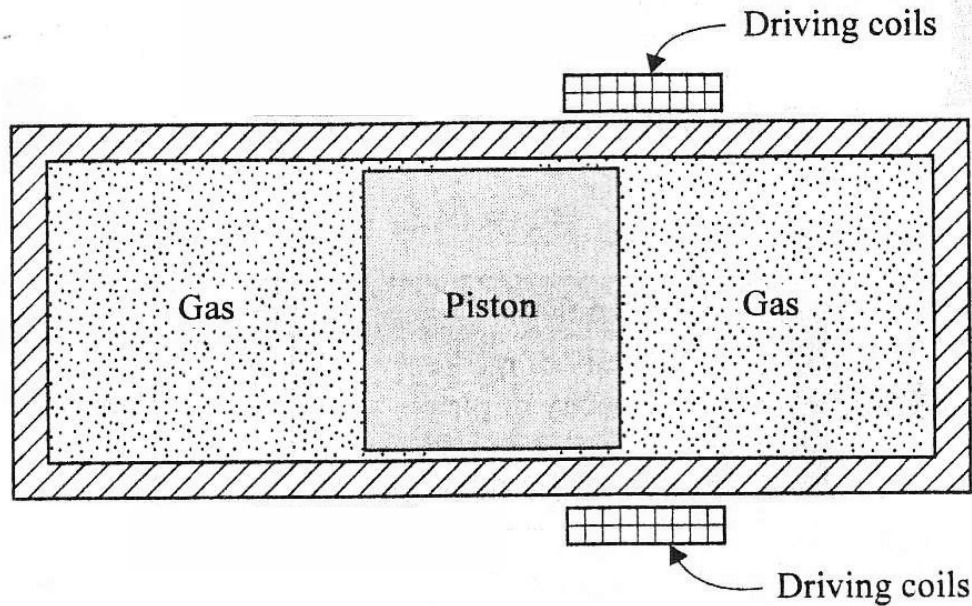


Figure 3: The second experimental setup used to measure  $\gamma$ .

drive coils are used to force oscillations of the piston at the frequency of the generator used to supply current to the coils. When the frequency of the generator is matched to the natural or resonant oscillation frequency  $f = 1/\tau$  the piston will have a large amplitude oscillations. When off resonance, the oscillation amplitude will be small. Measure the amplitude of the oscillations as a function of frequency. Take lots of data close to the resonance. Plot the oscillation amplitude as a

function of frequency and fit the data to a Lorentzian curve (just like you did for the *LRC* circuit of term 1). From the fit determine the peak frequency and the width of the resonance. From the peak frequency determine  $\gamma$  for air.

Note that the expression for the oscillation period  $\tau$  (Eq. 21) will be slightly modified for this second measurement. To see why, start with Eq. 19 which is repeated here after dividing by  $V^{\gamma-1}$ :

$$Vdp + \gamma p dV = 0. \tag{22}$$

Consider Fig. 4 which shows the cylinder initially in its equilibrium position (solid border lines) and after it has been displaced (dashed border lines). In Eq. 22,  $V$  refers to the volume on one side of

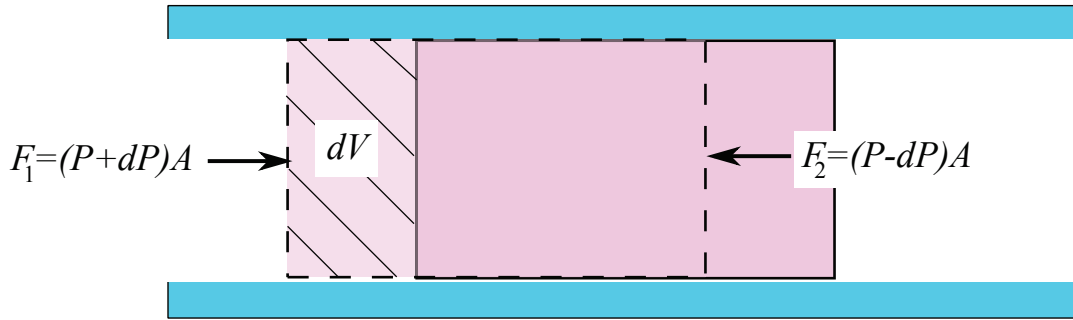


Figure 4: Movement of the piston causes changes in volume and pressure on both sides of the piston.

the cylinder and  $dV$  to the change in volume on one side of the cylinder. In the figure  $dV$  is shown as the shaded region and is given by  $dV = Ax$  where  $A$  is the cross-sectional area of the cylindrical tube and  $x$  is the displacement from the equilibrium position of the cylinder. Likewise,  $dp$  is the change in pressure on one side of the cylinder.

The net force acting on the cylinder is:

$$F = F_1 - F_2 = 2Adp, \tag{23}$$

such that:

$$dp = \frac{F}{2A} = \frac{ma}{2A} = \frac{m}{2A}\ddot{x}. \tag{24}$$

Note that the force is in the *opposite* direction of the displacement making this a *restoring* force. Substituting for  $dV$  and  $dp$  in Eq. 22 gives:

$$m\ddot{x} = -\frac{2\gamma p A^2}{V}x. \tag{25}$$

This is the same result as Eq. 20 except for the factor of 2 in the numerator. Realizing that this expression gives simple harmonic motion with frequency  $\omega = 2\pi f = \sqrt{k/m}$  with  $k = 2\gamma pA^2/V$  gives an oscillation period of:

$$\tau = 2\pi\sqrt{\frac{mV}{2\gamma pA^2}}. \quad (26)$$

This is the final result. This expression is valid when the piston is located in the centre of the tube and  $V$  is the volume on one side of the piston (i.e. approximately half the total volume of the tube).